Single-scale diagrams and multiple binomial sums.

M. Yu. Kalmykov[1], O. Veretin[2]

DESY Zeuthen, Theory Group, Platanenallee 6, D-15738, Zeuthen, Germany

Abstract

The ε-expansion of several two-loop self-energy diagrams with different thresholds and one mass are calculated. On-shell results are reduced to multiple binomial sums which values are presented in analytical form.

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[1] E-mail: kalmykov@ifh.de
On leave of absence from BLTP, JINR, 141980, Dubna (Moscow Region), Russia
[2] E-mail: veretin@ifh.de

1 Introduction

Substantial progress in multiloop Feynman diagram calculations in recent years requires computation of scalar master integrals. Often the problem involving different mass scales can be reduced (e.g. by expanding) to integrals depending only on a single scale. Thus single-scale diagrams (e.g. bubbles with one non-zero mass, massless self-energy, massive on-shell self-energy integrals, etc.) form an important class of Feynman diagrams. Such integrals arise for example in renormalization group calculations. The structure of massless integrals is well understood now [1, 2]. In particular recently a correspondence between knot theory and massless diagrams [3] has been found which can serve as a very useful guide to find the transcendental numbers which occur with rational coefficient in the counterterms. This relationship is known only for diagrams that are free of subdivergences.

The transcendental structure of massive single-scale diagrams is less investigated [4]-[12] (see also [13], [14]). In particular, we do not know whether there exist a theory to predict the transcendental numbers for these diagrams. Recently it was observed [11] that all two-loop massive on-shell diagrams of propagator type without subdivergences can be written in following way

$$m^2 \mathbf{I}_0 \,|_{p^2=m^2} = r_1 \zeta_3 + r_2 \pi \mathrm{Ls}_2\left(\frac{\pi}{3}\right) + r_3 i\pi \zeta_2 + \mathcal{O}(\varepsilon), \tag{1}$$

where $\zeta_a = \zeta(a)$ is the Riemann ζ-function, r_j are rational coefficients and definition of $\mathrm{Ls}_n(z)$ is given by (3). This observation suggests a conjecture that irrationalities occurring in these diagrams are defined by the topology of a diagram but not e.g. by the distribution of the masses on lines. In this paper we test this conjecture in the next order of the ε-expansion.

Another problem under consideration is the test of the hypothesis about the connection between transcendental numbers occurring in the ε-expansion of diagrams and the presence of certain massive-particles-cuts. This conjecture reads as follows: zero-, one- and three massive particle cuts give rise to appearance of structures $\pi^j \zeta_m (\ln 2)^n \mathrm{Li}_p(1/2)$, where $\mathrm{Li}_p(x)$ is polylogarithm, or more complicated structures associated with Euler–Zagier sums (or multidimensional zeta/harmonic sums) [2, 3, 8, 15, 16]

$$\zeta(a_1, \ldots, a_k) = \sum_{n_i > n_{i+1}} \prod_{j=1}^{k} \frac{(\mathrm{sign}\, a_j)^{n_i}}{n_i^{|a_j|}}, \tag{2}$$

whereas the two massive particle particle cuts bring other transcendental numbers connected with "sixth root of unity" [8, 12]: $\left(\frac{\pi}{\sqrt{3}}\right)^k \zeta_m (\ln 3)^n \mathrm{Ls}_p(z_i)\, \mathrm{Ls}_q^{(r)}(z_j)$, where $z_k = \left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ and $\mathrm{Ls}_n(z)$ and $\mathrm{Ls}_n^{(m)}(z)$ are so-called log-sine integrals [17] defined by

$$\mathrm{Ls}_n(\theta) = -\int_0^\theta \ln^{n-1}\left|2\sin\frac{\phi}{2}\right| d\phi, \qquad \mathrm{Ls}_n^{(m)}(\theta) = -\int_0^\theta \phi^m \ln^{n-m-1}\left|2\sin\frac{\phi}{2}\right| d\phi. \tag{3}$$

We will show here that some of these irrationalities are related to sums [18]-[21]

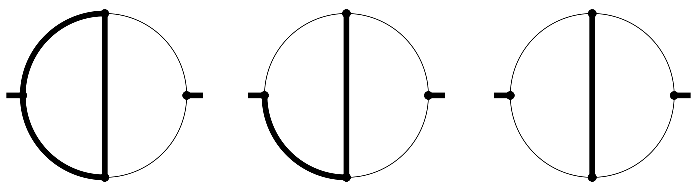
$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} \frac{1}{n^c} \prod_{a,b,i,j} \left[\sum_{m=1}^{n-1} \frac{1}{m^a}\right]^i \left[\sum_{k=1}^{2n-1} \frac{1}{k^b}\right]^j \tag{4}$$

which we call *multiple binomial sums.*

The question about transcendental structures connected with four- and more massive particle cuts remains open.

2 Results

As examples we consider the diagrams shown in Fig.1. All diagrams posses different cuts in the external variable p^2 with following values of thresholds $I_{125} = \{0, 4m^2\}$, $I_{15} = \{0, 1m^2, 4m^2\}$, and $I_5 = \{0, 1m^2\}$.



$\mathbf{I}_{125}$ $\mathbf{I}_{15}$ $\mathbf{I}_{5}$

Figure 1: Bold and thin lines correspond to massive and massless propagators, respectively.

To evaluate these diagrams we use the semianalytic method developed in Ref. [22]. This approach is based on a possibility to restore analytical results in terms of harmonic sums from several first coefficients of the small momentum expansion [24]. In Ref. [22] the $\mathcal{O}(1)$ parts of the diagrams shown in Fig.1 were found [3]. Here we extend these results calculating their ε-parts. Omitting all technical details that can be found in the above paper we present the results of our calculation[4]. We find ($z = p^2/m^2$)

$$\mathbf{I_{125}} = \frac{1}{p^2}\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}} z^n \left[\frac{-\ln(-z)}{n^2} + \frac{3}{n^3} + \varepsilon\left(\frac{\ln^2(-z)}{2n^2} - \ln(-z)\left\{\frac{2}{n^2} + \frac{S_1(n-1)}{n^2}\right\}\right.\right.$$
$$\left.\left. -\frac{1}{n^4} + \frac{6}{n^3} - \frac{\zeta_2}{n^2} + \frac{11}{n^3}S_1(n-1) - \frac{4}{n^3}S_1(2n-1)\right) + \mathcal{O}(\varepsilon^2)\right], \tag{5}$$

$$\mathbf{I_{15}} = \frac{1}{p^2}\sum_{n=1}^{\infty} z^n \left[-\frac{\ln(-z)}{n^2} - \frac{\zeta_2}{n} + \frac{2}{n^3} + \frac{3}{n}V_2(n-1) + \frac{1}{\binom{2n}{n}}\frac{4}{n^3}\right.$$
$$+\varepsilon\left(\frac{\ln^2(-z)}{2n^2} - \ln(-z)\left\{\frac{1}{n^3} + \frac{2}{n^2} + \frac{2}{n^2}S_1(n-1)\right\} + \frac{\zeta_3}{n} - \frac{2}{n}\zeta_2 + \frac{6}{n^3}S_1(n-1)\right.$$
$$-\frac{3}{n^2}V_2(n-1) + \frac{3}{n}V_3(n-1) + \frac{15}{n}V_{2,1}(n-1) - \frac{6}{n}\tilde{V}_{2,1}(n-1) + \frac{4}{n^3} + \frac{6}{n}V_2(n-1)$$

[3] The finite part of $\mathbf{I_5}$ is given in [25], $\mathbf{I_{125}}$ in [5] and exact result for $\mathbf{I_{125}}$ in terms of hypergeometric function presented in [26].

[4] We are working in Minkowski space-time with dimension $N = 4 - 2\varepsilon$. For each loop we assume a common normalization factor $(m^2 e^{\gamma})^{\varepsilon}/\pi^{\frac{N}{2}}$, where γ is Euler constant.

$$+\frac{1}{\binom{2n}{n}}\left\{\frac{8}{n^3}+\frac{20}{n^3}S_1(n-1)-\frac{8}{n^3}S_1(2n-1)\right\}\Bigg)+\mathcal{O}(\varepsilon^2)\Bigg], \qquad (6)$$

$$\mathbf{I_5} = \frac{1}{p^2}\sum_{n=1}^{\infty}(-z)^n\Bigg[-\frac{\ln^2(-z)}{n}+\frac{2}{n^2}\ln(-z)-\frac{2}{n}\zeta_2+\frac{4}{n}K_2(n-1)-\frac{2}{n^3}-2\frac{(-)^n}{n^3}$$
$$+\varepsilon\Bigg(\frac{\ln^3(-z)}{n}-\frac{\ln^2(-z)}{n}\left\{\frac{2}{n}+\frac{3}{n^2}+\frac{S_1(n-1)}{n}\right\}+\ln(-z)\left\{\frac{4}{n^2}+\frac{6}{n^3}+\frac{2}{n^2}S_1(n-1)\right.$$
$$\left.+\frac{2}{n}S_2(n-1)-\frac{2}{n}\zeta_2\right\}+\frac{2}{n}\zeta_3+\zeta_2\left\{\frac{2}{n^2}-\frac{4}{n}-\frac{2}{n}S_1(n-1)\right\}-\frac{4}{n^3}-\frac{6}{n^4}-\frac{2}{n^3}S_1(n-1)$$
$$-\frac{2}{n^2}S_2(n-1)-\frac{2}{n}S_3(n-1)+\frac{8}{n}K_2(n-1)+\frac{12}{n}K_{2,1}(n-1)+\frac{4}{n}K_2(n-1)S_1(n-1)$$
$$+\frac{2}{n}K_3(n-1)-(-)^n\left\{\frac{4}{n^3}+\frac{2}{n^4}+\frac{8}{n^3}S_1(n-1)\right\}\Bigg)+\mathcal{O}(\varepsilon^2)\Bigg], \qquad (7)$$

where we use the following notations for the finite sums elaborated in [22]

$$S_a(n)=\sum_{j=1}^{n}\frac{1}{j^a}, \qquad K_a(n)=-\sum_{j=1}^{n}\frac{(-)^j}{j^a}, \qquad K_{a,b}(n)=-\sum_{j=1}^{n}\frac{(-1)^j}{j^a}S_b(j-1),$$

$$V_a(n)=\sum_{j=1}^{n}\binom{2j}{j}^{-1}\frac{1}{j^a}, \quad V_{a,b}(n)=\sum_{j=1}^{n}\binom{2j}{j}^{-1}\frac{1}{j^a}S_b(j-1), \quad \tilde{V}_{a,b}(n)=\sum_{j=1}^{n}\binom{2j}{j}^{-1}\frac{1}{j^a}S_b(2j-1).$$

Sums K_a, $K_{a,b}$ were also used in [15]. Sums V_a and $\tilde{V}_a$ were predicted by differential equation method [23],

For all infinite series occurring in (5)-(7) a one-fold integral representation for arbitrary z can be written [22]. Thus these series can be continued analytically in the whole complex z-plane. Some of these integrals can be rewritten in terms of polylogarithms. For example we note the following representation

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{z^n}{n^a} &= \int_0^1\frac{ds}{s}S_{a-2,1}\big(zs(1-s)\big) \\
&= \frac{1}{(a-2)!}\int_0^{2\arcsin\frac{\sqrt{z}}{2}}\left[\ln z-2\ln\left(2\sin\frac{\theta}{2}\right)\right]^{a-2}\theta\,d\theta \\
&= -\sum_{j=0}^{a-2}\frac{(-2)^j}{(a-2-j)!j!}(\ln z)^{a-2-j}\,\mathrm{Ls}_{j+2}^{(1)}\left(2\arcsin\frac{\sqrt{z}}{2}\right), \qquad (8)
\end{aligned}$$

where $a>1$ and $S_{a,b}(z)$ are generalized Nielsen polylogarithms [27]. The last two lines have been obtained in [21]. Substituting $z=1$ into (8) (i.e. on-shell condition for Feynman diagram) we get

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n^a}\equiv V_a(\infty)=-\frac{(-2)^{a-2}}{(a-2)!}\mathrm{Ls}_a^{(1)}\left(\frac{\pi}{3}\right).$$

For $a = 1, \ldots, 5$ we can write explicitly [17]

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} \frac{1}{n^{\{1,2,3,4,5\}}} = \Bigg\{ \frac{1}{3}\frac{\pi}{\sqrt{3}}, \ \ \frac{1}{3}\zeta_2, \ \ \frac{2}{3}\pi \mathrm{Ls}_2\left(\frac{\pi}{3}\right) - \frac{4}{3}\zeta_3, \ \ \frac{17}{36}\zeta_4, \\ \frac{4}{9}\pi \mathrm{Ls}_4\left(\frac{\pi}{3}\right) - \frac{19}{3}\zeta_5 - \frac{2}{3}\zeta_2\zeta_3 \Bigg\}.$$

However, the analytical results for arbitrary z for other types of sums are not yet available. For example we may write the integral representation

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} \frac{z^n}{n^a} S_1(n-1) = \int_0^1 \frac{ds}{s} S_{a-2,2}\Big(zs(1-s)\Big) =$$
$$-\frac{2}{(a-2)!} \int_0^{2\arcsin\frac{\sqrt{z}}{2}} \left[\ln z - 2\ln\left(2\sin\frac{\theta}{2}\right)\right]^{a-2} \left[\mathrm{Ls}_2(\pi+\theta) + \theta \ln\left(2\sin\frac{\pi+\theta}{2}\right)\right] d\theta, \quad (9)$$

but we do not know how to evaluate these integrals explicitly for $a > 2$ even at $z = 1$. Nevertheless for each particular $a = 1, \ldots, 5$ we are able to obtain an analytical answer for (9) at $z = 1$ using the **PSLQ** algorithm [28]. This proceeds as follows. Each sum can be evaluated numerically with arbitrary accuracy. **PSLQ** expresses the obtained numerical value in terms of given transcendental numbers. The only problem is to define the full set of "basis" elements. Such a basis for diagrams having two massive particle cut was elaborated in [12]. The Ansatz for the construction of basis up to arbitrary order have been suggested and explicitly evaluated up to weight 5 that corresponds to the second order in ε-expansion of two-loop propagator type diagrams. The important role in construction of this basis belongs to Broadhurst's observations [8] that sixth root of unity plays an important role in the calculation of the diagrams.

We have investigated all sums of type (4) up to weight 5. Not all of them are expressible in terms of our basis elements [8, 12] or the transcendental numbers given in [2]. But it turns out that linear combinations of sums occurring in Feynman diagrams evaluated in the present paper are connected with basis given by the "sixth root of unity". All our results were obtained empirically by carefully compiling and examining a huge data base of high precision (several hundreds decimals) numerical calculations. Some details of this calculations are given in Appendix A. The results of multiple binomial sum's elaboration up to weight 4 are collected in Appendix B. The results for sum of weight 5 are relatively lengthy and therefore will not be published here.

We note that all V-type sums (occurring e.g. in (6)) are reduced to the multiple binomial sums due to the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^a} \sum_{j=1}^{n-1} f(j) = \sum_{n=1}^{\infty} f(n) \left[\zeta_a - S_a(n-1) - \frac{1}{n^a} \right].$$

Finally we mention that all sums occurring in Eq.(5) are expressible in terms of Euler-Zagier sums (2).

3 Conclusion

Collecting the results of Appendix B we obtain the following values for on-shell integrals

$$m^2\mathbf{I_{125}}\left.\right|_{p^2=m^2} = \left\{-4\zeta_3 + 2\pi\mathrm{Ls}_2\left(\frac{\pi}{3}\right) + i\pi\frac{\zeta_2}{3}\right\}(1+2\varepsilon)$$

$$+\varepsilon\left(\frac{16}{3}\left[\mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right]^2 - 7\pi\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) - \frac{488}{9}\zeta_4 - i\pi\left\{\frac{4}{9}\pi\mathrm{Ls}_2\left(\frac{\pi}{3}\right) - \frac{11}{9}\zeta_3\right\}\right) + \mathcal{O}(\varepsilon^2), \quad (10)$$

$$m^2\mathbf{I_{15}}\left.\right|_{p^2=m^2} = \left\{-3\zeta_3 + 2\pi\mathrm{Ls}_2\left(\frac{\pi}{3}\right) + i\pi\zeta_2\right\}(1+2\varepsilon)$$

$$+\varepsilon\left(6\left[\mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right]^2 - 9\pi\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) - \frac{2567}{36}\zeta_4 + i\pi\frac{5}{4}\zeta_3\right) + \mathcal{O}(\varepsilon^2), \quad (11)$$

$$m^2\mathbf{I_5}\left.\right|_{p^2=m^2} = \{-3\zeta_3 + i\pi\zeta_2\}(1+2\varepsilon)$$

$$+\varepsilon\left(6\zeta_2\ln^2 2 - \ln^4 2 - 24\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{57}{4}\zeta_4 + i\pi\left\{\frac{19}{4}\zeta_3 - 9\zeta_2\ln 2\right\}\right) + \mathcal{O}(\varepsilon^2). \quad (12)$$

It is convenient to multiply Eqs.(10)–(12) by $(1-2\varepsilon)$. Then we can write (10) and (11) in the form

$$m^2(1-2\varepsilon)\mathbf{I}\left.\right|_{p^2=m^2} = r_1\zeta_3 + r_2\pi\mathrm{Ls}_2\left(\frac{\pi}{3}\right) + r_3 i\pi\zeta_2$$

$$+\varepsilon\left(r_4\left[\mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right]^2 + r_5\pi\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + r_6\zeta_4 + i\pi\left\{r_7\pi\mathrm{Ls}_2\left(\frac{\pi}{3}\right) + r_8\zeta_3\right\}\right) + \mathcal{O}(\varepsilon^2), \quad (13)$$

with some rational numbers r_j. Both $\mathbf{I_{125}}$ and $\mathbf{I_{15}}$ have a threshold at $4m^2$ plus possible thresholds at $0m^2$ and $1m^2$. The above results suggest that all such diagrams have the form (13) where coefficients r_j depend on the distribution of the masses on lines while the basis (13) is defined by the topology alone.

If a diagram has no threshold at $4m^2$ then it is expressible in terms of Euler–Zagier sums. The three particle massive cuts lead to the appearance of a new structures like $\mathrm{Ls}_4^{(1)}\left(\frac{2\pi}{3}\right)$ [8, 10, 12] and some others.

Let us return to the results of Appendix B. One can write down a representation in terms of a hypergeometric sum

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{z^n}{n^a} = \frac{z}{2}\;{}_{a+1}F_a\left(\begin{array}{c}\{1\}_{a+1};\\ \frac{3}{2},\{2\}_{a-1};\end{array}\frac{z}{4}\right).$$

It is easy to see that multiple sums with nested harmonic summations S_a can be obtained from the generating function

$${}_{p+1}F_p\left(\begin{array}{c}\{1+a_i\}_{p+1};\\ \frac{3}{2}+b;\ \{2+c_i\}_{p-1};\end{array}\frac{1}{4}\right) \quad (14)$$

by expanding (14) in powers of a_i, c_j and b.

There are certain sums (see Appendix B) which cannot be expressed (polynomially) in terms of a basis connected with "sixth root of unity" [12] or the one given in [2]. We don't have an explanation for this phenomenon. However all linear combinations arising in the Taylor expansion of (14) are expressible in terms of our basis.
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A Multiple precision calculation of $\mathbf{Ls}_n(\theta)$ and $\mathbf{Ls}_n^{(1)}(\theta)$

For the purposes of **PSLQ** we need to evaluate functions $\mathrm{Ls}_n(\theta)$ and $\mathrm{Ls}_n^{(1)}(\theta)$ to very high accuracy (several hundreds of decimals). It is clear that the definitions (3) are not suitable for such numerical calculations. For example, one needs several hours of running MAPLE to calculate $\mathrm{Ls}_6(\pi/3)$ with accuracy about 200 decimals. As an alternative we found the following series for these functions which allows us obtain the results with needed accuracy in a few seconds. We have ($z = 4\sin^2(\theta/2)$)

$$\mathrm{Ls}_n(\theta) = (n-1)!\,\sqrt{z}\sum_{r=1}^{n}\frac{(-\log\sqrt{z})^{n-r}}{(n-r)!}\;\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^r}\left(\frac{z}{16}\right)^k,$$

$$\mathrm{Ls}_n^{(1)}(\theta) = -(-1)^n\frac{(n-2)!}{2^{n-2}}\sum_{r=0}^{n-2}\frac{(-\log z)^r}{r!}\;\sum_{k=1}^{\infty}\frac{z^k}{\binom{2k}{k}k^{n-r}}.$$

B Multiple binomial sums

In this section we present the results of our searching of relationships between multiple binomial sum[5] (4) up to weight 4 and the set of transcendental numbers given in [8, 12]. All sums were obtained numerically by using multiprecision FORTRAN with accuracy of about 300 decimals and analytical results were obtained by **PSLQ**. The sums of weight 3 can be extracted from results of [8, 22]. The sums of weigt 4 and depths $k < 3$ are investigated in [8].

Below we omit argument of harmonic sums implying that $S_a \equiv S_a(n-1)$ and $\bar{S}_a \equiv S_a(2n-1)$.

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}S_1 = -\frac{1}{3}\frac{\pi}{\sqrt{3}}\ln 3 + \frac{4}{3}\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}},$$

[5] All multiple binomial sums (4) can be rewritten in terms of function $\Psi(n) = \mathrm{d}/\mathrm{d}n\log\Gamma(n)$ and its derivatives by means of the following relation

$$\Psi^{(k-1)}(j) = (-)^k(k-1)!\,[\zeta_k - S_k(j-1)]\,, \qquad k > 1,$$

where $\Psi^{(k)}(z)$ is the k-th derivative of the Ψ-function. In particular, for $k=1$ we have $\Psi(j) = S_1(j-1) - \gamma$.

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n^2}S_1=-\frac{4}{9}\pi\mathrm{Ls}_2\left(\frac{\pi}{3}\right)+\frac{11}{9}\zeta_3,$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n^3}S_1=-\pi\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+\frac{4}{3}\left[\mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right]^2-\frac{269}{36}\zeta_4,$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}\bar{S}_1=-\frac{1}{3}\frac{\pi}{\sqrt{3}}\ln 3+\frac{7}{3}\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}},$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n^2}\bar{S}_1=-\frac{7}{9}\pi\mathrm{Ls}_2\left(\frac{\pi}{3}\right)+\frac{23}{9}\zeta_3,$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n^3}\bar{S}_1=-\pi\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+\frac{7}{3}\left[\mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right]^2-\frac{143}{18}\zeta_4,$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}S_1^2=\frac{1}{3}\frac{\pi}{\sqrt{3}}\ln^2 3-\frac{8}{3}\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}}\ln 3+\frac{55}{27}\frac{\pi}{\sqrt{3}}\zeta_2+4\frac{\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)}{\sqrt{3}},$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n^2}S_1^2=\frac{4}{3}\pi\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)-\frac{16}{9}\left[\mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right]^2+\frac{1085}{108}\zeta_4,$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}S_1^3=-\frac{1}{3}\frac{\pi}{\sqrt{3}}\ln^3 3+4\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}}\ln^2 3-\frac{55}{9}\frac{\pi}{\sqrt{3}}\zeta_2\ln 3$$

$$-12\frac{\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)}{\sqrt{3}}\ln 3+\frac{2}{3}\zeta_2\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}}-\frac{179}{27}\frac{\pi}{\sqrt{3}}\zeta_3-\frac{92}{27}\frac{\mathrm{Ls}_4\left(\frac{\pi}{3}\right)}{\sqrt{3}}+8\frac{\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)}{\sqrt{3}},$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}S_1\bar{S}_1=\frac{1}{3}\frac{\pi}{\sqrt{3}}\ln^2 3-\frac{11}{3}\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}}\ln 3+\frac{157}{54}\frac{\pi}{\sqrt{3}}\zeta_2+\frac{11}{2}\frac{\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)}{\sqrt{3}},$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n^2}S_1\bar{S}_1=\frac{11}{6}\pi\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)-\frac{28}{9}\left[\mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right]^2+\frac{3125}{216}\zeta_4,$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}S_1^2\bar{S}_1=-\frac{1}{3}\frac{\pi}{\sqrt{3}}\ln^3 3+5\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}}\ln^2 3-\frac{212}{27}\frac{\pi}{\sqrt{3}}\zeta_2\ln 3$$

$$-15\frac{\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)}{\sqrt{3}}\ln 3+\frac{8}{9}\zeta_2\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}}-\frac{727}{81}\frac{\pi}{\sqrt{3}}\zeta_3-\frac{298}{81}\frac{\mathrm{Ls}_4\left(\frac{\pi}{3}\right)}{\sqrt{3}}+10\frac{\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)}{\sqrt{3}},$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}S_2=\frac{1}{27}\frac{\pi}{\sqrt{3}}\zeta_2,$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n^2}S_2=\frac{5}{108}\zeta_4,$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}S_3 = -\frac{2}{3}\zeta_2\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}} - \frac{16}{9}\frac{\pi}{\sqrt{3}}\zeta_3 + \frac{4}{3}\frac{\mathrm{Ls}_4\left(\frac{\pi}{3}\right)}{\sqrt{3}},$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}S_1S_2 = -\frac{1}{27}\frac{\pi}{\sqrt{3}}\zeta_2\ln 3 - \frac{2}{9}\zeta_2\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}} + \frac{49}{81}\frac{\pi}{\sqrt{3}}\zeta_3 - \frac{20}{81}\frac{\mathrm{Ls}_4\left(\frac{\pi}{3}\right)}{\sqrt{3}},$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}S_2\bar{S}_1 = -\frac{1}{27}\frac{\pi}{\sqrt{3}}\zeta_2\ln 3 + \frac{4}{9}\zeta_2\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}} + \frac{229}{81}\frac{\pi}{\sqrt{3}}\zeta_3 - \frac{146}{81}\frac{\mathrm{Ls}_4\left(\frac{\pi}{3}\right)}{\sqrt{3}},$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}\left(\bar{S}_1^2+\bar{S}_2\right) = \frac{1}{3}\frac{\pi}{\sqrt{3}}\ln^2 3 - \frac{14}{3}\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}}\ln 3 + \frac{113}{27}\frac{\pi}{\sqrt{3}}\zeta_2 + 7\frac{\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)}{\sqrt{3}},$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n^2}\left(\bar{S}_1^2+\bar{S}_2\right) = \frac{7}{3}\pi\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) - \frac{49}{9}\left[\mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right]^2 + \frac{4505}{216}\zeta_4,$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}S_1\left(\bar{S}_1^2+\bar{S}_2\right) = -\frac{1}{3}\frac{\pi}{\sqrt{3}}\ln^3 3 + 6\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}}\ln^2 3 - 10\frac{\pi}{\sqrt{3}}\zeta_2\ln 3 - \frac{112}{9}\frac{\pi}{\sqrt{3}}\zeta_3$$
$$-18\frac{\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)}{\sqrt{3}}\ln 3 + \frac{2}{3}\zeta_2\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}} - \frac{8}{3}\frac{\mathrm{Ls}_4\left(\frac{\pi}{3}\right)}{\sqrt{3}} + 12\frac{\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)}{\sqrt{3}},$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{1}{n}\left(\bar{S}_1^3+3\bar{S}_1\bar{S}_2+2\bar{S}_3\right) = -\frac{1}{3}\frac{\pi}{\sqrt{3}}\ln^3 3 + 7\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}}\ln^2 3 - \frac{394}{27}\frac{\pi}{\sqrt{3}}\zeta_3$$
$$-\frac{113}{9}\frac{\pi}{\sqrt{3}}\zeta_2\ln 3 - 21\frac{\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)}{\sqrt{3}}\ln 3 + \frac{2}{3}\zeta_2\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}} - \frac{28}{27}\frac{\mathrm{Ls}_4\left(\frac{\pi}{3}\right)}{\sqrt{3}} + 14\frac{\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)}{\sqrt{3}}.$$

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